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CONVERGENCE OF POLYNOMIAL LEVEL SETS

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ABSTRACT. In this paper we give a characterization of pointwise and uniform convergence of sequences of homogeneous polynomials on a Banach space by means of the convergence of their level sets. Results are obtained both in the real and the complex cases, as well as some generalizations to the nonhomogeneous case and to holomorphic functions in the complex case.

Kuratowski convergence of closed sets is used in order to characterize pointwise convergence. We require uniform convergence of the distance function to get uniform convergence of the sequence of polynomials.

0. INTRODUCTION

Throughout this paper E will be a Banach space over $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$). $\mathcal{P}(^kE)$ will denote the space of k-homogeneous continuous polynomials on E, and $\mathcal{P}(E)$ the space of all continuous polynomials. (See [Ll] or [M] for a general reference on infinite dimensional polynomials.)

We define the norm on $\mathcal{P}(^kE)$ in the usual way as:

$$||P|| = Sup\{|P(x)| : ||x|| \leq 1\}.$$

Definition 0.1. Let X be a topological space, $cl(X)$ the closed subsets of X, $A_n, A \in cl(X) - \{\emptyset\}$. We say that $\{A_n\}$ is Kuratowski convergent (K-convergent) to A ($A_n \xrightarrow{K} A$) provided both of the following conditions hold:
(i) For each $a \in A$ there exists a sequence $\{a_n\}$ convergent to a such that $a_n \in A_n \ \forall n \in \mathbf{N}$
(ii) If J is a cofinal subset of $\mathbf{N}$ and $\{a_{n_k}\}_{k \in J}$ converges to a, where $a_{n_k} \in A_{n_k}$, then $a \in A$.

Condition (i) is equivalent to $A \subset LiA_n$, and (ii) is equivalent to $LsA_n \subset A$. See [K] for the definitions of LiA_n and LsA_n.

Definition 0.2. Let (X, d) be a metric space, $A, A_n \in cl(X) - \{\emptyset\}$. We say that $\{A_n\}$ is Wijsman convergent to A (W-convergent) ($A_n \xrightarrow{W} A$) if $d_n(x)$ converges pointwise to $d(x)$, where $d_n, d : X \to \mathbf{R}^+$ are defined as

$$d(x) = d(x, A), \quad d_n(x) = d(x, A_n).$$

Definition 0.3. Let (X, d) be a metric space, $A, A_n \in cl(X) - \{\emptyset\}$. We say that $\{A_n\}$ is r-convergent to A ($A_n \xrightarrow{r} A$) if $d_n(x)$ converges to $d(x)$ uniformly on bounded sets.

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Remark. (1) If (X,d) is a metric space, then W-convergence is weaker than r-convergence but stronger than K-convergence [B-P].

(2) As a consequence of the results included in this paper, we will prove that the three concepts are different. See also [B-P].

(3) In [B] it is proved that if X is a Banach space then r-convergence is equivalent to the following condition: $\forall r > 0, \forall \epsilon > 0$, there exists n_0 such that:

i) $A \cap rB \subset A_n + \epsilon B$,
ii) $A_n \cap rB \subset A + \epsilon B$,

$\forall n \geq n_0$, where B is the unit ball of the Banach space.

(4) If E is a finite dimensional Banach space then K,W and r-convergence agree [B-P].

(5) Definition 1 has meaning if some of the sets are empty.

(6) Definitions 2 and 3 can be extended to empty sets if we define $d(x,\emptyset) = +\infty$.

If $P \in \mathcal{P}(E)$ and $\alpha \in \mathbf{K}$ we use the following notation:

$$V(P-\alpha) = \{x \in E : P(x) = \alpha\}.$$

Definition 0.4. Let $\{P_n\}$ and P be continuous polynomials on E. We say that $\{P_n\}$ is Kuratowski convergent to P ($P_n \xrightarrow{K} P$, for short), respectively Wijsman convergent ($P_n \xrightarrow{W} P$), respectively r-convergent ($P_n \xrightarrow{r} P$) if $\{V(P_n - \alpha)\}$ is K-convergent, resp. W-convergent, resp. r-convergent to $V(P-\alpha)$ for all $\alpha \in \mathbf{K}$.

This definition has meaning if P_n, P are just continuous functions.

The following results are easily checked.

Proposition 0.1. *If $P, P_n \in \mathcal{P}(^kE)$ then $V(P-\alpha) = \alpha^{\frac{1}{k}}V(P-1)$ for all $\alpha \neq 0$ if $\mathbf{K} = \mathbf{C}$ or k is odd, and for all $\alpha > 0$ otherwise.*

Corollary 0.2. *If $P, P_n \in \mathcal{P}(^kE)$ then $\{V(P_n - \alpha)\}$ is Θ-convergent to $V(P-\alpha)$ for all $\alpha \neq 0$ if $\mathbf{K} = \mathbf{C}$ or k is odd, and for all $\alpha > 0$ otherwise, provided only that $\{V(P_n - 1)\}$ is Θ-convergent to $V(P-1)$. Θ=K,W,r.*

Using that $V(P-\alpha) = (-\alpha)^{\frac{1}{k}}V(P+1)$ if $\mathbf{K} = \mathbf{R}$, k even and $\alpha < 0$; we have:

Proposition 0.3. *If $P, P_n \in \mathcal{P}(^kE)$, $\Theta = K, W$ or r, then*
a) If $\mathbf{K} = \mathbf{C}$ or k is odd, then $\{P_n\}$ is Θ-convergent to P if and only if $\{V(P_n - 1)\}$ is Θ-convergent to $V(P-1)$ and $\{V(P_n)\}$ is Θ-convergent to $Vw(P)$.
b) Otherwise we also must check that $\{V(P_n+1)\}$ is Θ-convergent to $V(P+1)$, in order to have that $\{P_n\}$ is Θ-convergent to P.

Proposition 0.4. *Let P_n be in $\mathcal{P}(^{r_n}E)$, P_0 a continuous polynomial, and suppose that $\{P_n\}$ converges to P_0 pointwise. Then:*
a) If $\{r_n\}$ is unbounded then P_0 is identically 0.
b) If P_0 is not identically 0, then $\lim_n r_n = r_0$ (i.e. r_n is eventually r_0) and $P_0 \in \mathcal{P}(^{r_0}E)$.

Proof. We proceed with part a). Let $x \in E$. If $\{r_n\} \to +\infty$ (we may assume so without lost of generality) we have

$$P_0(2x) = \lim_n P_n(2x) = (\lim_n 2^{r_n})P_0(x).$$

And so necessarily $P_0(x) = 0$. Let's assume now that $\{r_n\}$ is bounded. If r_1 and r_2 are limit points of the sequence, then choosing an x such that $P_0(x) \neq 0$ and arguing as before, we have

$$P_0(2x) = 2^{r_1} P_0(x) = 2^{r_2} P_0(x) \text{ and therefore } \; r_1 = r_2.$$

Now, if r_0 is the limit of the sequence, then

$$P_0(\lambda x) = \lim_n P_n(\lambda x) = \lambda^{r_0} \lim_n P_n(x) = \lambda^{r_0} P_0(x).$$

We conclude that $P_0 \in \mathcal{P}(^{r_0}E)$ because $P_0 \in \mathcal{P}(E)$. □

1. K-convergence

In [B] the following result is proved:

Theorem 1.1. *If E is a real normed linear space, and $\{x_n^*\}$ is a sequence of nonzero linear functionals, then the following are equivalent:*
(1) $\{x_n^*\}$ is K-convergent to $x^* \in E - \{0\}$.
(2) $\{x_n^*\}$ is w^*-convergent to x^* and norm bounded.
(3) If $\{x_n\}$ is norm convergent to x, then $\lim_n x_n^*(x_n) = x^*(x)$.

The proof of this result uses linearity strongly. Our aim is to extend this result to the polynomical case.

We start with an analog of part a) of proposition 0.4, which we will use in the proof of proposition 1.6.

Proposition 1.2. *Let $P_n \in \mathcal{P}(^{r_n}E)$ for all $n \in \mathbf{N}$, P_0 a continuous function. $V(P_n - \alpha) \overset{K}{\to} V(P - \alpha)$ for all $\alpha \neq 0$ and $\{r_n\} \to +\infty$. Then P_0 is identically* 0.

Proof. Let x be such that $P_0(x) = \alpha$, $\alpha \neq 0$. There exists a sequence $\{x_n\}$ convergent to x such that $P_n(x_n) = \alpha$. Define $\epsilon_n = 2^{\frac{1}{r_n}}$; clearly ϵ_n goes to 1 and so $\{\epsilon_n x_n\}$ converges to x. But $\epsilon_n x_n \in V(P_n - 2\alpha)$ and therefore $x \in V(P - 2\alpha)$ by condition ii) in the definition of K-convergence. So we have $P(x) = 2\alpha$, contradicting the choice of x. □

Proposition 1.3. *Let f, f_n be continuous functions on a real Banach space E such that $f_n \overset{K}{\to} f$. Then $\lim_n f_n(x_n) = f(x)$ for every sequence $\{x_n\}$ converging to x.*

Proof. Let $\alpha = f(x)$. If the sequence $\{f_n(x_n)\}$ does not converge to α, we may assume, passing to a subsequence if necessary, that $f_n(x_n) > \alpha + \epsilon$ for all $n \in \mathbf{N}$, ϵ being a positive number. Now, using condition (i) of the definition of K-convergence, we choose a sequence $\{z_n\}$ converging to x such that $f_n(z_n) = \alpha$. By continuity of the f_n there exists y_n lying between x_n and z_n and hence converging to x such that $f_n(y_n) = \alpha + \frac{\epsilon}{2}$. Using condition (ii) of the definition, we have that $f(x) = \alpha + \frac{\epsilon}{2}$, contradicting the definition of α. □

Remarks. (1) The conclusion of the proposition is equivalent to uniform convergence on compact subsets.

(2) We have used the fact that $\mathbf{K} = \mathbf{R}$ just to get order in the functions range. So the result is true if we consider functions from a metric space with connected balls to $\mathbf{R}$. The following example proves that things are worse in the complex case.

Example 1.4. Let $w_n = e^{\frac{2\pi i}{n}}$ We define $f_n(w) = \phi_n(|w|)$, where

$$\phi_n(r) = \begin{cases} (1 - nr)w_n, & 0 \leq r \leq \frac{1}{n}, \\ 0, & \frac{1}{n} \leq r. \end{cases}$$

Then $f_n \stackrel{K}{\to} 0$, because if $\alpha \neq 0$, $V(f_n - \alpha)$ is nonempty for at most one n and $V(f_n)$ is K-convergent to $\mathbf{C}$. However $\{f_n(0)\}$ converges to 1.

If we denote the space of all holomorphic functions on E by $\mathcal{H}$(E), we have the following result in the complex case.

Proposition 1.5. *Let $f_n \in \mathcal{H}(E)$, f a continuous function. Suppose that*

$$\forall B \subset E \text{ bounded, there exists a positive constant } M \text{ such that}$$

$$||df_n(y)|| \leq M \quad \forall y \in B \quad \forall n.$$

If $f_n \stackrel{K}{\to} f$ then $\{f_n(x_n)\}$ converges to $f(x)$ for every sequence $\{x_n\}$ converging to x.

Proof. We denote $\alpha = f(x)$. By the definition of K-convergence there exists a sequence $\{z_n\}$ converging to x such that $f_n(z_n) = \alpha$. We have

$$|f_n(x_n) - f_n(z_n)| \leq ||df_n(y_n)||||x_n - z_n|| \leq M||x_n - z_n||$$

where y_n is a point lying between x_n and z_n, and so we may assume that $y_n \in B(0, ||x|| + 1)$. We conclude that $f_n(x_n) \to \alpha$. □

Let's note that in propositions 1.3 and 1.5 we only use $V(f_n - \alpha) \stackrel{K}{\to} V(f - \alpha)$, since $\alpha = f(x)$.

Proposition 1.6. *Let $P_n \in \mathcal{P}(^{r_n}E)$ and $P \in \mathcal{P}(^{r_0}E)$. If $V(P_n - \alpha) \stackrel{K}{\to} V(P - \alpha)$ for all $\alpha \neq 0$, then $\{P_n(x_n)\}$ converges to $P(x)$ whenever $\{x_n\}$ converges to x. Moreover, if P is not identically 0 then $\{||P_n||\}$ is bounded.*

Proof. First, let's prove the second part. If $\{||P_n||\}$ is not bounded, we may assume without lost of generality that $||P_n|| = c_n$ where $c_n \to +\infty$, or, better, that there exists a normalized sequence $\{x_n\}$ such that $P_n(x_n) = d_n$ with d_n going to $+\infty$. (In the real case, if necessary, we can replace P_n by $-P_n$ and P by $-P$). We denote now $d_n^{-\frac{1}{r_n}} x_n$ as z_n, and we have that $z_n \in V(P_n - 1)$, and, since $\{r_n\}$ is bounded (Proposition 1.2), $\lim_n z_n = 0$; so by the definition of K-convergence we conclude that $P(0) = 1$, which is not possible as P is homogeneous.

Let's denote $P(x) = \beta$, and $P_n(x_n) = \beta_n$. If $\beta = 0$ we have to prove that $\{\beta_n\}$ converges to 0. If this is not true we may assume that $|\beta_n| \geq \epsilon$ for a fixed positive number ϵ. Let $\alpha_n = \beta_n^{-\frac{1}{r_n}}$. The α_n are bounded, so passing to a subsequence we may assume they converge to a α. Now if we define z_n as $\alpha_n x_n$ we have that $z_n \in V(P_n - 1)$, but the sequence $\{z_n\}$ converges to αx and consequently $P(\alpha x) = 1$ ($V(P_n - 1) \stackrel{K}{\to} V(P - 1)$), contradicting the fact that $\beta = 0$. Small changes should be made in order to avoid problems in the real case—we probably have to consider $\alpha_n = (-\beta_n)^{-\frac{1}{r_n}}$, and $z_n \in V(P_n + 1)$.

If $\beta \neq 0$, proposition 1.3 gives us the result in the real case, and by proposition 1.5 we just have to prove that $\{||dP_n(y)||\}$ is uniformly bounded on bounded sets in order to get the complex case. But if A_n are the r_n-linear symmetric form associated to P_n, we have

$$||dP_n(y)|| \leq r_n||y||^{r_n - 1}||A_n|| \leq r_n||y||^{r_n - 1}\frac{(r_n)^{r_n}}{r_n!}||P_n||$$

and therefore $\{||dP_n(y)||\}$ is uniformly bounded on bounded sets because $\{r_n\}$ and $\{||P_n||\}$ are bounded. □

The above proposition together with Proposition 0.4 gives us the following alternative: either P is identically 0, or $r_n = r_0$ eventually. Therefore we may restrict our study to the following not exclusive cases: (a) $r_n = k \ \ \forall n$, (b) P identically 0. We will suppose too that $k \neq 0$, the constant case, being trivial

The following example shows us that when $P = 0$ we cannot claim that $\{||P_n||\}$ is bounded.

Example 1.7. Let E be an infinite-dimensional Banach space. By the Josefson-Nissenzweig Theorem [D] there exists a weak-star null sequence $\{x_n^*\}$ in the unit sphere of the dual E^*. We define $P_k(x) = k(x_k^*(x))^k$ and $P = 0$; each P_k belongs to $\mathcal{P}({}^kE)$ (moreover they are finite type polynomials). $||P_k|| = k$ We have $V(P_k) = Ker x_k^*$, and $\forall x \in E = V(P), \lim_n x_n^*(x) = 0$. On the other hand, $|x_k^*(x)| = d(x, Ker x_k^*)$ because $||x_k^*|| = 1$; hence there exists $z_k \in V(P_k)$ such that $\{z_k\}$ converges to x, and we conclude that $\{V(P_k)\}$ converges, in the Kuratowski sense, to $V(P)$.

Set $V(P-1) = \emptyset$ and $V(P_k - 1) = \{x \in E : x_k^*(x) = (\frac{1}{k})^{\frac{1}{k}}\}$. Then in order to prove that $P_n \xrightarrow{K} P$ we need to see that $\{x_k\}$ does not have convergent subsequences if each $x_k \in V(P_k - 1)$. But by the fact that $\{x_k^*\}$ is a bounded weak-star null sequence we deduce that if $\lim_n x_{k_n} = x$ then $\lim_n x_{k_n}^*(x_{k_n}) = 0$, which is not true because $x_{k_n}^*(x_{k_n}) = (\frac{1}{k_n})^{\frac{1}{k_n}}$.

In order to avoid problems with the roots in the real case, we may consider k odd.

If E is a finite dimensional Banach space, then compactness of closed bounded subsets give us convergence of the norm of the P_n to 0 if the sequence of polynomials is K-convergent to 0.

In order to say more about the behavior of the norm of K-convergent sequences of polynomials, we need the following proposition.

Proposition 1.8. *Let $P_n \in \mathcal{P}({}^{r_n}E)$, and P a continuous polynomial on E. If $V(P_n - \alpha) \xrightarrow{K} V(P - \alpha)$ for all $\alpha \neq 0$ and $\{x_n\} \subset E$ is bounded, then $\{|P_n(x_n)|^{\frac{1}{r_n}}\}$ is bounded too.*

Proof. We may assume that $P_n(x_n) = \alpha_n \neq 0 \ \ \forall n$. Let $\alpha \neq 0$; moreover, in the real case we will assume that $\alpha(\alpha_n)^{-1} > 0$. Then $(\frac{\alpha}{\alpha_n})^{\frac{1}{r_n}} x_n \in V(P_n - \alpha)$. If $\{|\alpha_n|^{\frac{1}{r_n}}\}$ is not bounded, then a subsequence of $\{(\frac{\alpha}{\alpha_n})^{\frac{1}{r_n}}\}$, and consequently a subsequence of $\{(\frac{\alpha}{\alpha_n})^{\frac{1}{r_n}} x_n\}$, goes to 0. K-convergence tells us that $0 \in V(P - \alpha)$, contradicting the fact that P is a homogeneus polynomial (Proposition 0.4). □

The following corollaries are now trivial:

Corollary 1.9. *If $P_n \in \mathcal{P}({}^{r_n}E)$, P is a continuous polynomial and $V(P_n - \alpha) \xrightarrow{K} V(P - \alpha)$ for all $\alpha \neq 0$, then $\{||P_n||^{\frac{1}{r_n}}\}$ is bounded.*

Corollary 1.10. *If $P_n, P \in \mathcal{P}({}^kE)$ and if $V(P_n - \alpha) \xrightarrow{K} V(P - \alpha)$ for all $\alpha \neq 0$, then $\{||P_n||\}$ is bounded.*

Example 1.7 proves that it is not true that K-convergence to 0 implies

$$\lim_n ||P_n||^{\frac{1}{r_n}} = 0 \ \ (||P_n||^{\frac{1}{r_n}} = n^{\frac{1}{n}} \to 1).$$

However we have the following converse.

Proposition 1.11. *Let $P_n \in \mathcal{P}({}^{r_n}E)$ and $P \in \mathcal{P}(E)$. If $V(P_n - \alpha) \xrightarrow{K} V(P-\alpha)$ for all $\alpha \neq 0$ and $\lim_n ||P_n||^{\frac{1}{r_n}} = 0$, then $P = 0$.*

Proof. We may assume $\{r_n\}$ is bounded. Let $x \in E$. There exists a sequence $\{x_n\}$ convergent to x, and therefore bounded by a positive constant M, such that $P_n(x_n) = P(x)$. Then

$$|P(x)| = |P_n(x_n)| \leq ||P_n||||x_n||^{r_n} \leq ||P_n||M^{r_n} = (||P_n||^{\frac{1}{r_n}}M)^{r_n}$$

which converges to 0 because $\{r_n\}$ is bounded. □

We are now going to study the converse of proposition 1.6. We have the following result:

Proposition 1.12. *Let $P_n \in \mathcal{P}({}^{r_n}E)$ and $P \in \mathcal{P}({}^{r_0}E)$. If for every sequence $\{x_n\}$ convergent to an arbitrary point $x \in E$ we have that $\{P_n(x_n)\}$ is convergent to $P(x)$, then $V(P_n - \alpha) \xrightarrow{K} V(P-\alpha) \quad \forall \alpha \neq 0$.*

Proof. By corollary 0.2, it is enough to prove the case $\alpha = 1$ (the proof when $\alpha = -1$, if necessary, is similar).

If P is not identically 0 and $x \in V(P-1)$, then the sequence $\{P_n(x)\}$ converges to 1, so we may assume that $P_n(x)$ is always diferent from 0 and consequently the sequence $\{x_n\}$, where $x_n = P_n(x)^{-\frac{1}{r_n}}x$, converges to x and $x_n \in V(P_n - 1)$.

On the other hand, if $\{x_{n_k}\}$ converges to x and $P_{n_k}(x_{n_k}) = 1$, clearly $P(x) = 1$.

Let's suppose now that $P = 0$; we have to see that $\{V(P_n - 1)\}$ converges to $\emptyset$. But if $x_{n_k} \in V(P_{n_k} - 1)$ then the sequence $\{x_{n_k}\}$ cannot be convergent, because if it were, the image by P of its limit would be 1. □

Joining the previous results, we conclude:

Theorem 1.13. *Let E be a Banach space over $\mathbf{K}$, and let $P_n \in \mathcal{P}({}^{r_n}E)$ and $P \in \mathcal{P}({}^{r_0}E)$. The following conditions are equivalent:*
i) $V(P_n - \alpha) \xrightarrow{K} V(P-\alpha) \quad \forall \alpha \neq 0$.
ii) $\lim_n P_n(x_n) = P(x)$ for every sequence $\{x_n\}$ converging to $x \in E$.
iii) $\{P_n\}$ converges to P uniformly on compact subsets of E.

Moreover, if we suppose that P, P_n are nonzero polynomials, then the conditions in the theorem are equivalent to
iv) $\{P_n\}$ converges to P pointwise and $\{||P_n||\}$ is bounded.

What about the convergence of $\{V(P_n)\}$? The different behavior of the case $\alpha = 0$ is related to the fact that 0 is the unique critical value of a homogeneus polynomial, and consequently it is possible to have a change of the topology of $V(P-\alpha)$ near $\alpha = 0$.

First we observe that if $P = 0$, we do not have the convergence of $\{V(P_n)\}$ to $V(P)$, even assuming strong conditions.

Example 1.14. $E = \mathbf{K}^2$, $P_n(x, y) = \frac{x}{n}$ are 1-homogeneous polynomials, $\lim_n ||P_n|| = 0$, but $\{V(P_n)\}$ does not converge (in the Kuratowski sense) to $V(0) = \mathbf{K}^2$, and consequently $\{P_n\}$ is not K-convergent to 0.

Theorem 1.15. *Let $f, f_n : E \to \mathbf{C}$ be nonconstant holomorphic functions such that $\lim_n f_n(x_n) = f(x)$ for every sequence $\{x_n\}$ convergent to $x \in E$. Then $f_n \xrightarrow{K} f$.*

Proof. We have that $LsV(f_n) \subset V(f)$, because if $f_{n_k}(x_{n_k}) = 0$ and $\{x_{n_k}\}$ converges then $f(\lim_k x_{n_k}) = 0$.

Let's prove that $V(f) \subset LiV(f_n)$. Let x be such that $f(x) = 0$. If there does not exist a sequence $\{x_n\}$ converging to x such that $f_n(x_n) = 0$, then we may assume that there exists an ϵ such that $V(f_n) \cap B(x, \epsilon) = \emptyset \ \ \forall n$. Now we consider a complex line $L = \{x + \lambda z : \lambda \in \mathbf{C}\}$ such that f is not identically 0 on L. Let h_0 and h_n denote the restriction to L of f and f_n respectively, and $\Omega = B(x, \epsilon) \cap L$. Then $h_n, h_0 : \Omega \to \mathbf{C}$ are 1-dimensional holomorphic functions, and $\{h_n\}$ converges uniformly to h_0 on Ω because $\{f_n\}$ converges uniformly to f on the compact $cl(\Omega)$.

The fact that h_n does not have zeros in Ω give us the following alternative (Hurwitz's Theorem): Either h_0 is identically 0 or it does not have zeros in Ω. Both are impossible by the choice of L and the fact that $h_0(x) = 0$.

To finish the proof we just have to note that $V(f_n - \alpha)$ is the set of zeros of the function $f_n - \alpha$, and clearly the hypotheses hold if we add a constant. □

Corollary 1.16. *If E is a complex Banach space, $P, P_n \in \mathcal{P}(^k E) - \{0\}$, $\{P_n\}$ converges pointwise to P and $\{||P_n||\}$ is bounded, then $\{V(P_n)\}$ converges to $V(P)$.*

We conclude that in the complex case we have the following global theorem.

Theorem 1.17. *Let E be a complex Banach space, P and P_n nonzero k-homogeneous polynomials. Then the following conditions are equivalent.*
i) $P_n \stackrel{K}{\to} P$.
ii) $V(P_n - 1) \stackrel{K}{\to} V(P - 1)$.
iii) $\{P_n(x_n)\}$ converges to $P(x)$ for every sequence $\{x_n\}$ converging to $x \in E$.
iv) $\{P_n\}$ converges to P pointwise and $\{||P_n||\}$ is bounded.

The following easy example shows us that in the real case things are worse.

Example 1.18. $E = \mathbf{R}^2$, $k = 2$, $P_n(x, y) = x^2 + \frac{1}{n} y^2$, $P(x, y) = x^2$. We have $||P_n|| = ||P|| = 1$, and $\lim_n P_n(x, y) = P(x, y)$ for all $(x, y) \in E$. But $\{V(P_n)\}$ does not converge to $V(P)$ in the Kuratowski sense, because $V(P_n) = \{(0, 0)\}$ and $V(P) = \{(0, y) : y \in \mathbf{R}\}$.

However we have:

Proposition 1.19. *Let E be a real Banach space, and let $P, P_n \in \mathcal{P}(^k E)$ be such that $dP(x) \neq 0 \ \ \forall x \neq 0$. If $\{P_n(x_n)\}$ converges to $P(x)$ for every sequence $\{x_n\}$ converging to $x \in E$, then $\{V(P_n)\}$ is K-convergent to $V(P)$.*

Proof. It is clear that $LsV(P_n) \subset V(P)$. To prove that $V(P) \subset LiV(P_n)$, we consider a point $x \in E$ such that $P(x) = 0$. If $x = 0$, then clearly $0 \in LiV(P_n)$; otherwise we choose y such that $dP(x)(y) \neq 0$ and define $f_n(r) = P_n(x + ry)$ and $f(r) = P(x + ry)$; f satisfies $f'(0) \neq 0$. But f_n and f are polynomials of degree k over $\mathbf{R}$ such that $\{f_n\}$ converges to f pointwise and consequently uniformly on bounded sets.

Now the fact that 0 is a root of f and $f'(0) \neq 0$ enables us to claim that there exists a sequence $\{\lambda_n\}$ of roots of the f_n converging to 0. So $x + \lambda_n y \in V(P_n)$ and $\lim_n (x + \lambda_n y) = x$, and the proof is finished. □

The same proof works for the following theorem.

Theorem 1.20. *If $P, P_n \in \mathcal{P}(E)$, and $\{P_n(x_n)\}$ converges to $P(x)$ for every sequence $\{x_n\}$ converging to $x \in E$, then $V(P_n - \alpha) \overset{K}{\to} V(P - \alpha)$ for every regular value α of P.*

2. W-Convergence

In [B] the following result is proved.

Proposition 2.1. *If $x_n^*, x^* \in E^* - \{0\}$, then $\{x_n^*\}$ is W-convergent to x^* if and only if it is weak-star convergent and $\lim_n ||x_n^*|| = ||x^*||$.*

The proof uses the formula

$$d(x, V(x^* - \alpha)) = \frac{|x^*(x) - \alpha|}{||x^*||}$$

which is not true for polynomials. However we have the following lemma.

Lemma 2.2. *If $P \in \mathcal{P}(^kE)$ where either k is odd or E is complex, then*

$$d(0, V(P - \alpha)) = (\frac{|\alpha|}{||P||})^{\frac{1}{k}}$$

if $\alpha \neq 0$ $\quad (d(0, V(P - 0)) = 0)$.

Proof. Since $|P(x)| \leq ||P|| ||x||^k$, we have

$$d(0, V(P - \alpha)) \geq (\frac{|\alpha|}{||P||})^{\frac{1}{k}}.$$

On the other hand, $\forall \epsilon > 0$ there exists x such that $|P(x)| > (||P|| - \epsilon)||x||^k$. Let's define $z = (\frac{\alpha}{P(x)})^{\frac{1}{k}} x$. Then $z \in V(P - \alpha)$ and

$$||z|| = |\frac{\alpha}{P(x)}|^{\frac{1}{k}} ||x|| < (\frac{|\alpha|}{||P|| - \epsilon})^{\frac{1}{k}}$$

and hence

$$d(0, V(P - \alpha)) \leq (\frac{|\alpha|}{||P|| - \epsilon})^{\frac{1}{k}} \qquad \forall \epsilon > 0.$$

If $\epsilon \to 0^+$ we obtain the other inequality, and the result holds. □

The lemma is not true when k is even and E is real, as the following example shows.

Example 2.3. $E = \mathbf{R}^2$, $P(x, y) = \frac{1}{4}y^2 - x^2$ and $\alpha = \frac{1}{4}$. Then $d(0, v(P - \alpha)) = 1$, but $(\frac{1/4}{1})^{\frac{1}{2}} = \frac{1}{2}$.

Proposition 2.4. *If P and P_n are homogeneous polynomials, $P_n \overset{K}{\to} P$ and $B \subset E$ is an open ball, then $cl(P(B)) \subset Licl(P_n(B))$.*

Proof. Let $r \in P(B)$. There exists $x \in B$ such that $P(x) = r$. K-convergence says that there exists a sequence $\{x_n\}$ converging to x such that $P_n(x_n) = r$; moreover we may assume that it is contained in B. Therefore $r \in P_n(B) \quad \forall n$, and consequently $r \in Licl(P_n(B))$. Since $Licl(P_n(B))$ is closed, we conclude that $cl(P(B)) \subset Licl(P_n(B))$. □

Remarks. (1) Looking at the proof, it is easy to realize that continuity of the P_n and P is enough, and that the result is also true if B is any ball.

(2) If $P \neq 0$ it suffices to suppose $V(P_n - \alpha) \xrightarrow{K} V(P - \alpha)\quad \forall \alpha \neq 0$, because we could prove $P(B) - \{0\} \subset Licl(P_n(B))$ and $cl(P(B)) = cl(P(B) - \{0\})$ as a consequence of the fact that if $0 \in P(B)$ it is not isolated because $P \neq 0$.

Kuratowski convergence is not enough to insure the equality, as the next example shows.

Example 2.5. Let $E = c_0$, $P_n(x) = e_1^*(x)^2 - e_n^*(x)^2$, $P(x) = e_1^*(x)^2$. P_n converges pointwise to P, $||P|| = 1$ and $||P_n|| = 1$ ($\mathbf{K} = \mathbf{R}$) or 2 ($\mathbf{K} = \mathbf{C}$). Therefore we have that $P_n \xrightarrow{K} P$ in the complex case (Theorem 1.17), and in the real case we need just to prove that $\{V(P_n)\}$ is K-convergent to $V(P)$.

i) If $x \in V(P)$ then $e_1^*(x) = 0$, so we define $x_n = x - e_n^*(x)e_n$, and it is easy to see that $P_n(x_n) = 0$ and $||x_n - x|| \leq |e_n^*(x)| \to 0$ when $n \to +\infty$.

ii) If $\lim_j x_{n_j} = x$ and $x_{n_j} \in V(P_{n_j})$, we have

$$||x_{n_j} - x|| \geq |e_1^*(x_{n_j}) - e_1^*(x)| = |e_1^*(x) \underline{+} e_{n_j}^*(x_{n_j})| \geq |e_1^*(x)| - |e_{n_j}^*(x_{n_j})|.$$

The central equality is a consequence of $P_{n_j}(x_{n_j}) = 0$. Hence

$$\begin{aligned} |e_1^*(x)| &\leq ||x_{n_j} - x|| + |e_{n_j}^*(x_{n_j})| \\ &\leq ||x_{n_j} - x|| + |e_{n_j}^*(x_{n_j} - x)| + |e_{n_j}^*(x)| \leq 2||x_{n_j} - x|| + |e_{n_j}^*(x)|. \end{aligned}$$

The limit of the last expression is 0 when n goes to $+\infty$, and consequently $e_1^*(x) = 0$; hence $x \in V(P)$. Therefore $P_n \xrightarrow{K} P$ in both the real and complex cases.

Now let $B = B(e_1, \frac{1}{2})$. In the real case $cl(P(B)) = [\frac{1}{4}, \frac{9}{4}]$ and $cl(P_n(B)) = [0, \frac{9}{4}]$, and consequently we do not have $Lscl(P_n(B)) \subset cl(P(B))$. In the complex case $P(B) = \{w^2 : w \in D\}$ where $D = D(1, \frac{1}{2})$. On the other hand $P_n(B) = h(A)$, where $h : \mathbf{C^2} \to \mathbf{C}$ is defined as $h(z_1, z_2) = z_1^2 - z_2^2$ and

$$A = \{(z_1, z_2) \in \mathbf{C^2} : z_1 \in D(1, \tfrac{1}{2}), z_2 \in D(0, \tfrac{1}{2})\}.$$

Clearly $Lscl(P_n(B)) \not\subset cl(P(B))$, because $[0, \frac{5}{2}] \subset Re(P_n(B))\quad \forall n$, but $[0, \frac{5}{2}] \not\subset Re(P(B))$.

Remarks. (1) Later on, we wil use this example to prove that $\{P_n\}$ is not W-convergent to P.

(2) In the real case example 2.5 says that $\lim_n ||P_n||_B = ||P||_B$, and K-convergence is not enough to prove $\lim_n cl(P_n(B)) = cl(P(B))$. In particular, it proves that

$$\lim_n P_n(x) = P(x) \forall x \text{ and } \lim_n ||P_n|| = ||P|| \not\Rightarrow \lim_n cl(P_n(B)) = cl(P(B)).$$

Proposition 2.6. *If P, P_n are homogeneous polynomials and $P_n \xrightarrow{K} P$, then $LsP_n(K) \subset P(K)$ for every compact subset K of E.*

Proof. If $r \in LsP_n(K)$ then there exists a sequence $\{r_{n_j}\}$ converging to r and such that $r_{n_j} \in P_{n_j}(K)$. Hence there exists a sequence $\{x_{n_j}\}$ in K such that $\lim_j P_{n_j}(x_{n_j}) = r$. But since K is compact there exists a subsequence convergent to an $x \in K$ (let's denote it $\{x_{n_j}\}$ again), and $P(x) = \lim_j P_{n_j}(x_{n_j}) = r$ by Proposition 1.6. Therefore $r \in P(K)$. ☐

Proposition 2.7. *Let $P, P_n \in \mathcal{P}(^kE)$. If $V(P_n - \alpha) \xrightarrow{W} V(P - \alpha)$ for all $\alpha \neq 0$, then $cl(P(B)) = \lim_n cl(P_n(B))$ for every ball $B \subset E$.*

Proof. If $P = 0$, the condition $V(P_n - \alpha) \xrightarrow{W} V(P - \alpha)$ means that $\{V(P_n - 1)\}$ W-converges to $\emptyset$, or equivalently $\lim_n d(x, V(P_n - 1)) = +\infty$ for all $x \in E$; in particular $\lim_n d(0, V(P_n - 1)) = +\infty$. It is easy to see that then $\lim_n ||P_n|| = 0$ and therefore $\lim_n cl(P_n(B)) = \{0\} = cl(P(B))$ for every ball $B \subset E$, since $\{P_n\}$ converges uniformly to 0 on B.

We suppose now that $P \neq 0$. Since W-convergence implies K-convergence, proposition 2.4 (see remarks) says that we have just to prove $Lscl(P_n(B)) \subset clP(B))$, or equivalently that if $w_{n_j} \in cl(P_{n_j}(B))$ and $w = \lim_j w_{n_j}$ then $w \in cl(P(B))$. We may assume without loss of generality that $w_n \in P_n(B)$ and $\lim_n w_n = w$. Let $B = B(x_0, r_0)$.

A. Case $w \neq 0$. If $w \notin cl(P(B))$, we claim that

$$\text{there exists } \ \eta > 0 \ \text{ such that } \ d(x_0, V(P - w)) > r_0 + \eta.$$

Let's prove the claim.

Let ϵ be such that $w \notin \{z : d(z, cl(P(B))) \leq \epsilon\}$ Let $z \in B(x_0, r_0 + \eta)$, and let $y_z \in B$ be such that $||y_z - z|| < \eta$. We have

$$|P(z) - P(y_z)| \leq M\eta$$

where M depends on r_0, k, and $||P||$. If we choose η such that $M\eta < \epsilon$, our work is done.

Thanks to the claim, W-convergence enables us to suppose that

$$d(x_0, V(P_n - w)) > r_0 + \eta \ \ \forall n.$$

We now choose $z_n \in B$ such that $P_n(z_n) = w_n$, and define $\theta_n = (\frac{w}{w_n})^{\frac{1}{k}}$. We have $\theta_n z_n \in V(P_n - w)$, and hence $d(x_0, V(P_n - w)) \leq ||x_0 - \theta_n z_n||$. Therefore we have the following contradiction:

$$r_0 + \eta \leq \lim_n d(x_0, V(P_n - w)) \leq \lim_n ||x_0 - \theta_n z_n|| \leq r_0$$

since

$$||x_0 - \theta_n z_n|| \leq ||x_0 - z_n|| + |1 - \theta_n|||z_n|| < r_0 + |1 - \theta_n|(||x_0|| + r_0)$$

and $\theta_n \to 1$.

B. Case $w = 0$. If $0 \notin cl(P(B))$ then there exists a positive ϵ such that $D(0, \epsilon) \cap P(B) = \emptyset$. Let $y_0 \in B$, and let $y_n \in B$ be such that $P_n(y_n) = w_n$. Since $\{w_n\}$ converges to 0, we may assume that $|w_n| < \frac{\epsilon}{2}$. Choose $z_n \in [y_0, y_n] \subset B$ such that $P_n(z_n) \in D(0, \epsilon) - D(0, \frac{\epsilon}{2})$ (we are using that $\lim_n P_n(y_0) = P(y_0)$, which is a consequence of K-convergence). $\{P_n(z_n)\}$ has a subsequence which converges to a $w_0 \in D(0, \epsilon) - \{0\}$, and using part A we get a contradiction. □

We now fix $x \in E$, $\alpha \in \mathbf{K}$, and define

$$\lambda = d(x, V(P - \alpha)), \quad \lambda_n = d(x, V(P_n - \alpha)) \quad \text{where } \ P, P_n \in \mathcal{P}(^kE) - \{0\}.$$

Since K-convergence is weaker than W-convergence, $P_n \xrightarrow{K} P$ does not imply $\lim_n \lambda_n = \lambda$. However we have the following result:

Proposition 2.8. *If $P_n \xrightarrow{K} P$, then $\forall \epsilon > 0$ there exists n_0 such that*

$$\lambda_n < \lambda + \epsilon \qquad \forall n \geq n_0.$$

Proof. Let $z \in B(x, \lambda + \epsilon)$ be such that $P(z) = \alpha$. K-convergence gives us a sequence $\{z_n\}$ converging to z such that $P_n(z_n) = \alpha$, so there exists an n_0 such that $z_n \in B(x, \lambda + \epsilon) \ \ \forall n \geq n_0$, and therefore $\lambda_n \leq ||z_n - x|| < \lambda + \epsilon$. ☐

Corollary 2.9. *If* $P_n \overset{K}{\to} P$*, then* $\limsup \lambda_n \leq \lambda$.

We study now the converse of proposition 2.7.

Lemma 2.10. *Let* $\alpha \neq 0$*. If* $Lscl(P_n(B)) \subset cl(P(B))$ *for every ball* $B \subset E$*, then* $\lambda \leq \liminf \lambda_n$

Proof. If the conclusion does not hold, then there exist $\lambda^* < \lambda$ and a subsequence of $\{\lambda_n\}$ (we denote it $\{\lambda_n\}$ again) such that $\lambda_n < \lambda^* \quad \forall n$. Let $B = B(x, \lambda^*)$. For every n there exists a $z_n \in B$ such that $P_n(z_n) = \alpha$; consequently $\alpha \in P_n(B) \ \ \forall n$, and therefore $\alpha \in cl(P(B))$. Now for every j we choose a $y_j \in B$ such that $|\alpha - P(y_j)| < \frac{1}{j}$ and define $\gamma_j = (\frac{\alpha}{P(y_j)})^{\frac{1}{k}}$ (note that since $\alpha \neq 0$, we may assume that $P(y_j) \neq 0$ and in the real case $sg(P(y_j)) = sg\alpha)$. Then $P(\gamma_j y_j) = \alpha$, and we claim that $\gamma_j y_j \in B(x, \frac{\lambda + \lambda^*}{2})$ eventually. So we obtain a contradiction with the definition of λ.

The claim is clear, since

$$||\gamma_j y_j - x|| \leq |\gamma_j - 1| ||y_j|| + ||y_j - x|| < |\gamma_j - 1|(||x|| + \lambda^*) + \lambda^* < \frac{\lambda + \lambda^*}{2}$$

when j is big enough. ☐

Proposition 2.11. *Let* $P, P_n \in \mathcal{P}(^k E)$*. If for every ball* $B \in E$ *we have that* $\lim_n cl(P_n(B)) = cl(P(B))$*, then* $V(P_n - \alpha) \overset{W}{\to} V(P - \alpha)$ *for all* $\alpha \neq 0$.

Proof. When $P = 0$ the result follows easyly from the fact that if $\lim_n cl(P_n(B)) = \{0\}$ for every ball B, then $\lim_n d(x, V(P_n - \alpha)) = +\infty$ for every $x \in E$ and $\alpha \neq 0$.

On the other hand, we may assume without lost of generality that no P_n is identically 0, because if infinite polynomials P_n were identically 0 then $cl(P_n(B)) = \{0\}$ for infinitely many values of n, and consequently $P = 0$.

Let $\alpha \neq 0$ and fix $x_0 \in E$. We have to prove that $\lim_n \lambda_n = \lambda$. By lemma 2.10, we know that $\lambda \leq \liminf \lambda_n$, so we just need to prove that $\forall \epsilon > 0$ there exists an n_0 such that $\lambda_n < \lambda + \epsilon \ \ \forall n \geq n_0$. Let $B = B(x_0, \lambda + \frac{\epsilon}{2})$. By definition of λ, there exists $z \in B$ such that $P(z) = \alpha$, and therefore $\alpha \in P(B)$. Since $P(B) \subset Licl(P_n(B))$, we may choose a sequence $\{\alpha_n\}$ converging to α such that $\alpha_n \in P_n(B)$. Hence there exists a sequence $\{z_n\}$ in B such that $P_n(z_n) = \alpha_n$. Let's define $\tilde{z}_n = (\frac{\alpha}{\alpha_n})^{\frac{1}{k}}$. Then $\tilde{z}_n \in V(P_n - \alpha)$ and $\tilde{z}_n \in B(x_0, \lambda + \epsilon)$ for n big enough, since $||\tilde{z}_n - z_n|| = |1 - (\frac{\alpha}{\alpha_n})^{\frac{1}{k}}| ||z_n||$. Therefore we conclude that $\lambda_n < \lambda + \epsilon$. ☐

Joining the previous results, we have

Theorem 2.12. *If* $P, P_n \in \mathcal{P}(^{\mathrm{k}}\mathrm{E})$*, then* $V(P_n - \alpha) \overset{W}{\to} V(P - \alpha)$ *for every nonzero* $\alpha \in \mathbf{K}$ *if and only if* $\lim_n cl(P_n(B)) = cl(P(B))$ *for every ball* $B \subset E$.

Using theorem 1.13 and theorem 2.12, we have

Proposition 2.13. *If* $P, P_n \in \mathcal{P}(^k E)$ *and* $V(P_n - \alpha) \overset{W}{\to} V(P - \alpha)$ *for every nonzero* $\alpha \in \mathbf{K}$*, then* $\lim_n P_n(x) = P(x)$ *for every* $x \in E$*, and* $\lim_n ||P_n|| = ||P||$.

Example 2.5 proves that the converse is not true, and also as gives us an example of a K-convergent sequence which is not W-convergent.

We now study the W-convergence of $\{V(P_n)\}$. We begin with the complex case.

Proposition 2.14. *Let E be a complex Banach space, P and P_n nonzero k-homogeneous polynomials. If $\lim_n cl(P_n(B)) = cl(P(B))$ for every open ball $B \subset E$, then $\lim_n \lambda_n(x) = \lambda(x) \ \ \forall x \in E$, where $\lambda_n(x) = d(x, V(P_n))$ and $\lambda(x) = d(x, V(P))$.*

Proof. By theorem 1.17 $P_n \xrightarrow{K} P$, since $\{P_n\}$ converges pointwise to P and $\{||P_n||\}$ is bounded. Then by corollary 2.9 $\limsup \lambda_n(x) \leq \lambda(x)$, and consequently if the result does not hold then there exists x such that

$$\liminf \lambda_n(x) < \lambda(x) = \lambda.$$

Passing to a subsequence if necessary, we may assume that $\lim_n \lambda_n(x) = \lambda^* < \lambda' < \lambda$. Let $x_0 \in B(0,1)$ be such that $P(x_0) \neq 0$. Then $0 \in P_n(B(x, \lambda'))$ for every n, and therefore $0 \in cl(P(B(x, \lambda')))$. This allows us to choose a sequence $\{y_n\}$ in $B(x, \lambda')$ such that $\lim_n P(y_n) = 0$.

Let $\epsilon = \lambda - \lambda'$ and define $\phi_n : D(0, \epsilon) \to \mathbf{C}$ as $\phi_n(w) = P(y_n + wx_0)$. A subsequence of $\{\phi_n\}$ converges uniformly to a 1-dimensional polynomial ϕ such that $\phi(0) = \lim_n \phi_n(0) = \lim_n P(y_n) = 0$. On the other hand, ϕ_n does not vanish, because P does not have zeros on $B(x, \lambda)$ by the definition of λ. Therefore using Hurwitz's Theorem we conclude that ϕ must be identically 0, contradicting the fact that $P(x_0) \neq 0$. ☐

Remarks. (1) If $P = 0$ the result is false, as the following easy example proves: $P, P_n : \mathbf{C} \to \mathbf{C}$, $P_n(z) = \frac{z}{n}$. We have that $\lambda(x) = 0$ and $\lambda_n(x) = |x|$, but $P(B) = 0$, $\ P_n(B) = \frac{1}{n}B$, and hence $\lim_n cl(P_n(B)) = cl(P(B))$ for any ball B.

(2) We may use the same proof to extend the result for nonconstant continuous polynomials, because from the condition

$$\lim_n cl(P_n(B)) = cl(P(B)) \ \text{ for every open ball } \ B \subset E$$

it follows that $\{P_n(x_n)\}$ converges to $P(x)$ for every sequence $\{x_n\}$ converging to x. So replacing theorem 1.17 by theorem 1.15 we get the result.

Proposition 2.14 is false in the real case even if we have stronger hypotheses.

Example 2.15. Let $P, P_n : c_0 \to \mathbf{R}$ be defined as

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2}(x_k - x_{k+1})^2, \qquad P_n(x) = \sum_{k=1}^{n} \frac{1}{k^2}(x_k - x_{k+1})^2.$$

$\{P_n\}$ converges uniformly on bounded sets to P and $P(x) = 0 \ \Leftrightarrow \ x = 0$ and

$$V(P) = \{0\}, \qquad V(P_n) = \{x \in c_0 : x_1 = \cdots = x_{n+1}\}.$$

Therefore we have that $d(e_1, V(P)) = 1$ and $d(e_1, V(P_n)) = \frac{1}{2}$, and consequently $\lim_n d(e_1, V(P_n)) \neq d(e_1, V(P))$. Obviously $\lim_n cl(P_n(B)) = cl(P(B))$, since $\{P_n\}$ is uniformly convergent to P.

Incidentally, example 2.15 proves that the following result, which is true in the complex case, does not hold in the real case:

$$\text{“}B = B(x_0, r), 0 \in cl(P(B)), \eta > 0 \Rightarrow V(P) \cap B(x_0, r + \eta) \neq \emptyset\text{”}.$$

It is enough to consider $x_0 = e_1, r = \frac{1}{2}, \eta = \frac{1}{4}$. Then it is clear that $0 \notin P(B(e_1, \frac{1}{2} + \frac{1}{4}))$ but $0 \in cl(P(B))$ since $z_n = (\frac{1}{2} + \frac{1}{n})e_1 + \sum_{j=2}^n (\frac{1}{2} - \frac{1}{n})e_j \in B$ and $0 \leq P(z_n) \leq \frac{5}{n^2} \to 0$.

Remark. $dP(x) = 0 \Leftrightarrow x = 0$ in example 2.15; then we can use proposition 1.19 to prove that $\{V(P_n)\}$ is K-convergent to $V(P)$ and therefore, using Theorem 2.12, $P_n \stackrel{K}{\to} P$.

Stronger conditions on P give us a similar result in the real case, namely:

Proposition 2.16. *Let E a real Banach space, and P and P_n continuous polynomials such that P satisfies the following condition:*

$$(*) \qquad \lim_n ||dP(y_n)|| = 0 \quad \Rightarrow \quad 0 \in \overline{co}(\{y_n\}).$$

If $\lim_n cl(P_n(B)) = cl(P(B))$ for every open ball $B \subset E$ then $\lim_n \lambda_n(x) = \lambda(x)$ $\forall x \in E$, where $\lambda_n(x) = d(x, V(P_n))$ and $\lambda(x) = d(x, V(P))$.

Proof. From the condition on the polynomial P we deduce that $dP(x) = 0$ if and only if $x = 0$, and therefore $P_n \stackrel{K}{\to} P$. As in proposition 2.14, if $\lim_n \lambda_n(x) \neq \lambda(x)$ we would have $\{y_n\} \subset B(x, \lambda')$ such that $\lim_n P(y_n) = 0$. Now we define

$$\phi_n : [-\epsilon, \epsilon] \to \mathbf{R}, \quad \phi_n(t) = P(y_n + tz_n)$$

where z_n are points in the unit ball such that $|dP(y_n)(z_n)| > \frac{1}{2}||dP(y_n)||$ These polynomials never vanish and converge to a polynomial ϕ such that $\phi(0) = 0$. Then necessarily $\phi'(0) = 0$ and $\lim_n dP(y_n)(z_n) = \lim_n \phi_n'(0) = 0$, so we conclude that $\lim_n ||dP(y_n)|| = 0$, which is not possible because $0 \notin B(x, \lambda)$ and $\overline{co}(\{y_n\}) \subset B(x, \lambda)$. □

Let's remark that condition $(*)$ on the polynomial P is weaker than the property of being a separating polynomial.

It is interesting to observe that in 2.12 we cannot replace balls by arbitrary bounded sets, because, W-convergence being a metric property, we may consider an equivalent norm such that W-convergence is different, but bounded sets agree. See [B-V].

3. R-convergence

The aim of this section is to prove the following theorem, which extends a similar result in the linear case. See [B].

Theorem 3.1. *If $P, P_n \in \mathcal{P}(^kE)$, then $\{P_n\}$ converges to P uniformly on bounded sets if and only if $V(P_n - \alpha) \stackrel{r}{\to} V(P - \alpha)$ for every $\alpha \neq 0$.*

We start with the sufficiency of r-convergence level sets. As in proposition 2.7, the result holds if $P = 0$, so we must prove the following proposition.

Proposition 3.2. *Let $P, P_n \in \mathcal{P}(^kE) - \{0\}$. If $V(P_n - \alpha) \stackrel{r}{\to} V(P - \alpha)$ for every $\alpha \neq 0$, then $\{P_n\}$ converges to P uniformly on bounded sets.*

Proof. If the conclusion does not hold, then there exist an $\epsilon > 0$, a ball $B \subset E$ and a sequence $\{x_n\}$ in B such that $|P(x_n) - P_n(x_n)| > \epsilon$ (pass to a subsequence if necessary). Let $P(x_n) = \alpha_n$; since $\{\alpha_n\}$ is bounded we may assume that it converges to α (passing to a subsequence again).

Case $\alpha \neq 0$. Without lost of generality we may assume that $\frac{1}{2} < |\frac{\alpha}{\alpha_n}| < 2$ ($\frac{1}{2} < \frac{\alpha}{\alpha_n} < 2$ in the real case). We define $\lambda(x)$ and $\lambda_n(x)$ as above, and r-convergence says that $\{\lambda_n\}$ converge uniformly on bounded sets to λ.

Let's define $\tilde{x}_n = (\frac{\alpha}{\alpha_n})^{\frac{1}{k}} x_n$, $\tilde{x}_n \in 2B$. We have

$$(*) \qquad |\alpha - P_n(\tilde{x}_n)| = |\frac{\alpha}{\alpha_n}(P_n(x_n) - P(x_n))| > \frac{\epsilon}{2}.$$

On the other hand $|\lambda_n(\tilde{x}_n)| = |\lambda_n(\tilde{x}_n) - \lambda(\tilde{x}_n)|$, and therefore (by uniform convergence of $\{\lambda_n\}$ on $2B$)

$$\forall \eta > 0 \text{ there exists } n_0 \text{ such that } |\lambda_n(\tilde{x}_n)| < \eta \quad \forall n \geq n_0.$$

We may write this condition as $\forall\eta$ there exists $\tilde{z}_n \in V(P_n - \alpha)$ such that $||\tilde{z}_n - \tilde{x}_n|| < \eta$. Hence

$$|\alpha - P_n(\tilde{x}_n)| = |P_n(\tilde{z}_n) - P_n(\tilde{x}_n)| \leq M\eta < \frac{\epsilon}{2}$$

if we start with $\eta < \frac{\epsilon}{2M}$, where M is a constant depending only on B, k and $||P_n||$ which is uniformly bounded by Proposition 1.6. This contradicts $(*)$.

Case $\alpha = 0$. This means that $\lim_n P(x_n) = 0$, and consequently $|P_n(x_n)| > \frac{\epsilon}{2}$ eventually. Since $\{P_n(x_n)\}$ is bounded, we may assume that $\lim_n P_n(x_n) = \beta$ with $\beta \neq 0$. If we denote $\beta_n = P_n(x_n)$, we may assume as above that $\frac{1}{2} < |\frac{\beta}{\beta_n}| < 2$ ($\frac{1}{2} < \frac{\beta}{\beta_n} < 2$ in the real case). Hence

$$|P(\tilde{x}_n) - \beta| = |\frac{\beta}{\beta_n}(P(x_n) - P_n(x_n))| > \frac{\epsilon}{2}$$

where $\tilde{x}_n = (\frac{\beta}{\beta_n})^{\frac{1}{k}} x_n \in 2B \cap V(P_n - \beta)$. Now if we define

$$d(x) = d(x, V(P - \beta)) \quad \text{and} \quad d_n(x) = d(x, V(P_n - \beta))$$

we have that there exists n_0 such that

$$|d(\tilde{x}_n)| = |d(\tilde{x}_n) - d_n(\tilde{x}_n)| < \frac{\epsilon}{2M} \quad \forall n \geq n_0$$

since $d_n(\tilde{x}_n) = 0$ and $\{d_n\}$ converges uniformly to d on $2B$, where M is a constant depending only on B, k and $||P||$ that we will fix later. Hence there exists $\tilde{z}_n \in V(P - \beta)$ such that $||\tilde{z}_n - \tilde{x}_n|| < \frac{\epsilon}{2M}$, and consequently

$$|P(\tilde{x}_n) - \beta| = |P(\tilde{x}_n) - P(\tilde{z}_n)| < M\frac{\epsilon}{2M} = \frac{\epsilon}{2}$$

which is not possible. ☐

Now we prove the converse, and once again the case $P = 0$ is dealt with as for W-convergence.

Proposition 3.3. *Let $P, P_n \in \mathcal{P}(^kE) - \{0\}$. If $\{P_n\}$ converges to P uniformly on bounded sets, then $V(P_n - \alpha) \xrightarrow{r} V(P - \alpha)$ for every $\alpha \neq 0$.*

Proof. Let's fix $\alpha \neq 0$ and define the functions λ and λ_n as in proposition 3.2. We have to prove that $\{\lambda_n\}$ converge to λ uniformly on bounded sets. In order to achieve it, we fix $B = B(0, r)$ and $\epsilon \in (0, 1)$. From the definition of λ we get

$$(*) \qquad \forall x \in B \text{ there exists } z_x \in V(P - \alpha) \text{ such that } ||x - z_x|| < \lambda(x) + \frac{\epsilon}{2}.$$

Let $\tilde{B} = B(0, R)$, where $R = 3r + \lambda(0) + 1$. Then $z_x \in \tilde{B}$, since

$$||z_x|| < ||x|| + \lambda(x) + \frac{\epsilon}{2} < r + \lambda(x) + 1 \leq r + 2r + \lambda(0) + 1 = R.$$

Let's define $\alpha_n(x) = P_n(z_x)$. Since $\alpha \neq 0$ and $|\alpha_n(x) - \alpha| = |P_n(z_x) - P(z_x)|$, uniform convergence on $\tilde{B}$ of $\{P_n\}$ to P gives us $\lim_n \frac{\alpha}{\alpha_n(x)} = 1$ uniformly on B (we may assume that $\alpha_n(x)$ never vanishes). If we now define $z_n(x)$ as $(\frac{\alpha}{\alpha_n(x)})^{\frac{1}{k}} z_x$, then we have

$$(**) \qquad ||z_n(x) - z_x|| = |1 - (\frac{\alpha}{\alpha_n(x)})^{\frac{1}{k}}|||z_x|| < \frac{\epsilon}{2} \quad \forall n \geq n_0(\epsilon).$$

Using $(*)$ and $(**)$, we obtain $\lambda_n(x) - \lambda(x) < \epsilon$ since

$$\begin{aligned}\lambda_n(x) - \lambda(x) &< \lambda_n(x) - ||x - z_x|| + \frac{\epsilon}{2} \\ &\leq [\lambda_n(x) - \|x - z_n(x)\|] + \|z_n(x) - z_x\| + \frac{\epsilon}{2} \\ &\leq \|z_n(x) - z_x\| + \frac{\epsilon}{2} < \epsilon \quad \forall n \geq n_0(\epsilon) \text{ and } \forall x \in B.\end{aligned}$$

The other inequality holds in a similar way: let's define $z_{n,x} \in V(P_n - \alpha)$ such that $||x - z_{n,x}|| < \lambda_n(x) + \frac{\epsilon}{2}$. Since $\lim_n \lambda_n(0) = \lambda(0)$ we may assume $z_{n,x} \in \tilde{B}$. If we denote $\alpha_n(x) = P(z_{n,x})$ and $\tilde{z}_{nx} = (\frac{\alpha}{\alpha_n(x)})^{\frac{1}{k}} z_{n,x}$ and choose $\eta > 0$ such that $|(\frac{t}{t_n})^{\frac{1}{k}} - 1| < \frac{\epsilon}{2R}$ whenever $|t - t_n| < \eta$, we find that there exists $n_0(\epsilon)$ such that $|\alpha - \alpha_n(x)| = |P_n(z_{n,x}) - P(z_{n,x})| < \eta$ for all $n \geq n_0(\epsilon)$ (by uniform convergence of $\{P_n\}$ on $\tilde{B}$), and consequently

$$\begin{aligned}\lambda(x) - \lambda_n(x) &< \lambda(x) - ||x - z_{n,x}|| + \frac{\epsilon}{2} \\ &\leq \lambda(x) - ||\tilde{z}_{nx} - x|| + ||\tilde{z}_{nx} - z_{n,x}|| + \frac{\epsilon}{2} \\ &\leq ||\tilde{z}_{nx} - z_{n,x}|| + \frac{\epsilon}{2} = |(\frac{\alpha}{\alpha_n(x)})^{\frac{1}{k}} - 1|||z_{n,x}|| + \frac{\epsilon}{2} < \epsilon\end{aligned}$$

for every $x \in B$ and $n \geq n_0(\epsilon)$.

We conclude that $\lim_n \lambda_n(x) = \lambda(x)$ uniformly on B. □

From propositions 3.2 and 3.3 we conclude the validity of theorem 3.1. Our last step will be the study of the case $\alpha = 0$.

Proposition 3.4. *If E is a complex Banach space and P, P_n are nonzero k-homogeneous polynomials such that $\{P_n\}$ converges uniformly on bounded sets to P, then the sequence $\{V(P_n)\}$ is r-convergent to $V(P)$.*

Proof. We will use remark 3 in definition 0.3. Let B denote the unit ball of E.

First we will prove that $\forall r, \epsilon > 0$ there exists n_0 such that

$$V(P_n) \cap rB \subset V(P) + \epsilon B \quad \forall n \geq n_0.$$

Indeed otherwise there would exist $r, \epsilon > 0$ and a sequence $\{x_n\}$ contained in rB such that $P_n(x_n) = 0$ but $B(x_n, \epsilon) \cap V(P) = \emptyset$. Now uniform convergence on rB of $\{P_n\}$ gives us $\lim_n P(x_n) = 0$.

Let's choose $z_0 \in B$ such that $P(z_0) \neq 0$ and define $\phi_n : D(0,\epsilon) \to \mathbf{C}$ as $\phi_n(w) = P(x_n + wz_0)$. Then

$$\phi_n(x) = P(x_n) + \binom{k}{1} A(x_n,\dots,x_n,z_0)w + \cdots + \binom{k}{k-1} A(x_n,z_0,\dots,z_0)w^{k-1} + P(z_0)w^k$$

where A denotes the k-linear form associated to P. A subsequence of $\{\phi_n\}$ converges to a polynomial ϕ such that $\phi^{k)}(0) = k!P(z_0) \neq 0$. But $\phi(0) = \lim_n \phi_n(0) = \lim_n P_n(x_n) = 0$, and this, together with the fact that ϕ_n does not vanishs, gives us (by means of Hurwitz's Theorem) that $\phi = 0$, which is a contradiction.

We proceed with the other inclusion: $\forall r, \epsilon > 0$ there exists n_0 such that

$$V(P) \cap rB \subset V(P_n) + \epsilon B \quad \forall n \geq n_0$$

If this is not the case, then there exist $r > 0$ and $\epsilon > 0$ and $\{x_n\}$ be such that $x_n \in rB$, $P(x_n) = 0$ and $V(P_n) \cap B(x_n,\epsilon) = \emptyset$. Let $z_0 \in B$ be such that $P(z_0) \neq 0$, and define $\phi_n, \tilde{\phi}_n : D(0,\epsilon) \to \mathbf{C}$ as

$$\phi_n(w) = P(x_n + wz_0), \qquad \tilde{\phi}_n(w) = P_n(x_n + wz_0).$$

$\tilde{\phi}_n$ never vanishes. We choose now a subsequence of $\{\phi_n\}$ converging to a polynomial ϕ identically zero, such that $\phi(0) = 0$; and it is clear that the corresponding subsequence of $\{\tilde{\phi}_n\}$ converges to ϕ too. Hurwitz's Theorem again gives us the contradiction. □

Example 2.15 proves that in the real case uniform convergence on bounded sets of $\{P_n\}$ does not imply r-convergence (even W-convergence) of $\{V(P_n)\}$ to $V(P)$.

The following example shows us that, even assuming W-convergence, r-convergence of the 0-level sets does not follow from uniform convergence on bounded sets (in the real case) of the sequence $\{P_n\}$

Example 3.5. $E = c_0$, $P_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} x_k^2$, $P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} x_k^2$. $\{P_n\}$ converges uniformly to P, and

$$V(P) = \{0\}, \qquad V(P_n) = \{x \in c_0 : x_1 = \dots x_n = 0\}.$$

We have $\lim_n \lambda_n(x) = \lambda(x)$ for every $x \in E$, since $\lambda(x) = ||x||$, and $\lambda_n(x) = ||x||$ too when $n > n_0$, where n_0 is such that $|x_n| < \frac{1}{2}||x|| \; \forall n \geq n_0$. However $V(P_n) \cap 2B$ is never included in $V(P) + \frac{1}{2}B$, since $e_{n+1} \in V(P_n) \cap 2B$ but $e_{n+1} \notin \frac{1}{2}B$ for every n.

Let's observe that $dP(x) = 2\sum_{k=1}^{\infty} \frac{1}{k^2} x_k e_k^*$ and therefore

$$||dP(x)||_1 = 2\sum_{k=1}^{\infty} \frac{1}{k^2}|x_k|;$$

hence $\inf\{||dP(x)||_1 : x \in S\} = 0$. The last fact suggests the following result.

Proposition 3.6. *Let E be a real Banach space, P and P_n k-homogeneous polynomials such that* $\text{Inf}\{||dP(x)|| : ||x|| = 1\} > 0$. *If* $\lim_n P_n = P$ *uniformly on bounded sets, then $\{V(P_n)\}$ is r-convergent to $V(P)$.*

Proof. As in proposition 3.4, we will use the characterization set in Remark (3) of §0.3.

First we will prove that, if B is the unit ball of E, then $\forall r, \epsilon > 0$ there exists n_0 such that

$$V(P_n) \cap rB \subset V(P) + \epsilon B \quad \forall n \geq n_0.$$

Otherwise there exist $r, \epsilon > 0$ and a sequence $\{x_n\}$ contained in rB such that $P_n(x_n) = 0$ but $B(x_n, \epsilon) \cap V(P) = \emptyset$. Now uniform convergence on rB of $\{P_n\}$ gives us $\lim_n P(x_n) = 0$.

Since $x_n \in rB - \epsilon B \; \forall n$, we have a positive constant c such that $||dP(x_n)|| > c \; \forall n$, and we may choose y_n in the unit sphere such that $|dP(x_n)(y_n)| > c \;\; \forall n$. Let's define $\varphi_n : (-\epsilon, \epsilon) \to \mathbf{R}$ as $\varphi_n(r) = P(x_n + r y_n)$ A subsequence of $\{\varphi_n\}$ converges to a polynomial φ such that $\varphi(0) = \lim_n \varphi_n(0) = \lim_n P(x_n) = 0$, amd the fact that φ_n does not vanish gives us that $\varphi'(0) = 0$ or equivalently that $\lim_n dP(x_n)(y_n) = 0$ which is a contradiction.

We are now going to prove the other inclusion: $\forall r, \epsilon > 0$ there exists n_0 such that

$$V(P) \cap rB \subset V(P_n) + \epsilon B \quad \forall n \geq n_0.$$

If this is not so, then there exist $r > 0$ and $\epsilon > 0$ and $\{x_n\}$ such that $x_n \in rB$, $P(x_n) = 0$ and $V(P_n) \cap B(x_n, \epsilon) = \emptyset$. Considering $\{y_n\}$ as in the other inclusion, we may define $\phi, \phi_n : (-\epsilon, \epsilon) \to \mathbf{R}$ as

$$\phi(r) = \lim_n P(x_n + r y_n), \qquad \phi_n(r) = P_n(x_n + r y_n)$$

passing to a subsequence if necessary; ϕ_n never vanishes, and $\{\phi_n\}$ converges to ϕ because of the uniform convergence of the sequence $\{P_n\}$ to P on $(r + \epsilon)B$; but $\phi(0) = 0$ and consequently $\phi'(0) = 0$ too, and so is clear that $\lim_n dP(x_n)(y_n) = 0$, which gives us the contradiction. □

Let's observe that the condition $\mathrm{Inf}\{||dP(x)|| : ||x|| = 1\} > 0$ is weaker than the property of being a separating polynomial but strictly stronger than property $(*)$ in proposition 2.16, as the polynomial P in 3.5 proves.

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